**NUMERICAL METHODS**

**APPROXIMATE SOLUTIONS TO EQUATIONS**

**AIM**
- To find the approximate solution to complicated equations

**HOW DO YOU SOLVE THESE EQUATIONS?**

1. **$x^2 - 3x = 0$**
   - **Method?** Factorise
   - $x(x - 3) = 0$
   - $x = 0$ and $x = 3$
   - **Graph**
   - **U-shaped**
   - **Roots = 0 and 3**

2. **$x^2 - 3x - 4 = 0$**
   - **Method?** Factorise
   - $(x - 4)(x + 1) = 0$
   - $x - 4 = 0$ and $x + 1 = 0$
   - $x = 4$ and $x = -1$
   - **Graph**
   - **U-shaped**
   - **Roots = 4 and -1**

3. **$x^2 - 6x - 3 = 0$**
   - **Method?** Quadratic Formula
   - $a = 1$, $b = -6$, $c = -3$
   - $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
   - $x = \frac{6 \pm \sqrt{36 + 12}}{2}$
   - $x = 6.46$ and $x = -0.464$
\[ x^2 - 4x - 3 = 0 \]
\[ (x - 3)^2 - 12 = 0 \]
\[ (x - 3)^2 = 12 \]
\[ x - 3 = \pm \sqrt{12} \]
\[ x = 3 \pm \sqrt{12} \]
\[ x = 6.46 \text{ and } x = -0.464 \]

Graph
U-shaped
Roots = 6.46 and -0.464

Method?
Completing Square

\[ x^3 - 3x^2 + 5 = 0 \]

Method?
We can do sketch but we won’t be able to find out the roots

Our aim is to find an approximate value of this root!

NUMERICAL METHODS
CHANGE OF SIGN
ITERATION METHODS

WHAT IS A CHANGE OF SIGN?
Prove that there's a root that lies between 0 and 2:
\[ y = x^3 + 2x^2 - 3 \]

There must be a line that crosses the x-axis between 0 and 2.

Yes, there is a root between 0 and 2 because there is a change of sign.

**WHEN TO USE THIS METHOD**

- **ONLY** to **PROVE** that there's a solution that lies between two values.
- **We DO NOT CALCULATE** the root.

**HOW TO USE THIS METHOD**

- Rearrange the equation into the form \( f(x) = 0 \) or \( y = 0 \).
- Sketch the graph of the function (optional).
- Substitute the two values where the root lies.
- Show that there is a **change of sign**.

**Example:**

\[ f(x) = 6 + 3x - 2x^2 - 3x^3 \]

There must be a line that crosses the x-axis between 1 and 2.

Yes, there is a root between 1 and 2 because there is a change of sign.

**Example 1:** By sketching a suitable graph, show that the equation \( f(x) = x^3 - 3x^2 + 5 = 0 \) has only one root and that this root lies between -2 and -1.

Find: (i) Minimum point (ii) Maximum point
Example 1: By sketching a suitable graph, show that the equation \( f(x) = x^3 - 3x^2 + 5 = 0 \) has only one root and that this root lies between -2 and -1.

\[ f(x) = x^3 - 3x^2 + 5 \]

As \( x = -2 \),
\[ f(-2) = (-2)^3 - 3(-2)^2 + 5 = -8 - 12 + 5 = -15 \]

As \( x = 1 \),
\[ f(-1) = (-1)^3 - 3(-1)^2 + 5 = -1 - 3 + 5 = 1 \]

Yes, there is a root between -2 and -1 because there is a change of sign.

Example 2 (i): By sketching a suitable pair of graphs, show that the equation \( 2 \cot x = 1 + e^x \) where \( x \) is in radians, has only one root in the interval \( 0 < x < \frac{\pi}{2} \).

\[ 2 \cot x = 1 + e^x \]

\[ y = 2 \cot x - 1 - e^x \]

Refer to Grapher

Example 2(ii): Verify by calculation that this root lies between 0.5 and 1.0.

\[ 2 \cot x = 1 + e^x \]

\[ 2 \cot x - 1 - e^x = 0 \]

\( x \) is in radians!

At \( x = 0.5 \),
\[ y = 2 \cot(0.5) - 1 - e^{0.5} = \frac{2}{0.5} - 1 - e^{0.5} = -1.93 \]

At \( x = 1.0 \),
\[ y = 2 \cot(1) - 1 - e^{1} = \frac{2}{1} - 1 - e = -2.43 \]

Yes, there is a root between 0.5 and 1.0 because there is a change of sign.

HOMEWORK

- By sketching a pair of graphs, show that the equation \( \cos 2x = 1 - x \) has only one root which lies between \( 0 < x < \frac{\pi}{2} \) where \( x \) is in radians.
- Verify that this root lies between 0.5 and 0.75.
- By sketching a pair of graphs, show that the equation \( \sec x = 2 - x^2 \) has only one solution which lies between \( 0 < x < \frac{\pi}{2} \) where \( x \) is in radians.
- Proof that this solutions lies between 0.5 and 1.

NUMERICAL METHODS

ITERATION METHOD

- An iterative process is a repetition procedure designed to produce a sequence of approximation to some numerical quantity.
ITERATION FORMULA

• Choose the correct $x$ to be the subject
• Change the subject $x$ into $x_{n+1}$
• Change the rest of the $x$ into $x_n$

(i) $3x^2 = x - 1$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$x_n$</th>
<th>$x_{n+1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>$x_n$</td>
<td>$x_n - 1$</td>
</tr>
<tr>
<td>$n+1$</td>
<td>$x_{n+1}$</td>
<td>$3x_{n+1}^2 + 1$</td>
</tr>
</tbody>
</table>

(ii) $\sin 2x = x - 3$

<table>
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<th>$x_{n+1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>$x_n$</td>
<td>$\sin^{-1}(x_n - 3)$</td>
</tr>
<tr>
<td>$n+1$</td>
<td>$x_{n+1}$</td>
<td>$\frac{1}{2}(x_{n+1} - 3)$</td>
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</table>

(iii) $x^3 = 1 - x^2$

<table>
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<tr>
<th>$n$</th>
<th>$x_n$</th>
<th>$x_{n+1}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>$x_n$</td>
<td>$\sqrt{1 - x_n^2}$</td>
</tr>
<tr>
<td>$n+1$</td>
<td>$x_{n+1}$</td>
<td>$\sqrt{1 - x_{n+1}^2}$</td>
</tr>
</tbody>
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HOW TO CHOOSE THE CORRECT X?

• USE TRIAL AND ERROR (the iteration must not be fluctuating)

Example 3(a): Show that the equation $xe^x = 1$ has a root in the interval $0.55 < x < 0.65$

• change of sign

$xe^x = 1$

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y = xe^x - 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$0.55$</td>
<td>$-0.0467$</td>
</tr>
<tr>
<td>$0.65$</td>
<td>$0.245$</td>
</tr>
</tbody>
</table>

Yes, there is a root between 0.55 and 0.65 because there is a change of sign
Example 3(b): Use an iterative method to solve the equation $xe^x = 1$ with initial approx. $x_0 = 0.55$

- choose the correct iteration formula
  \[ x_{n+1} = \ln \left( \frac{1}{x_n} \right) \]

- show by calculation that this root lies between $-1$ and $0$

Example 4(ii): Show that if a sequence of values given by the iterative formula
  \[ x_{n+1} = \frac{2x_n^3 - 1}{3x_n^2 + 1} \]

converges, then it converges to the root of the equation given in part (i)

- get the original equation back
  \[ x^3 + x + 1 = 0 \]

- sketch graphs
  (i) $y = \cosec x$
  (ii) $y = \frac{1}{2}x + 1$

Refer to Grapher
Example 5(ii): Verify by calculation that this root lies between 0.5 and 1.

- change of sign

\[ \csc x = \frac{1}{2}x + 1 \]
\[ \csc x - \frac{1}{2}x - 1 = 0 \]

At \( x = 0.5 \),
\[ y = \csc 0.5 - \frac{1}{2}(0.5) - 1 = -0.312 \]
Yes, there is a root between 0.5 and 1 because there is a change of sign.

Example 5(iii): Show that this root also satisfies the equation

\[ x = \sin^{-1} \left( \frac{1}{2} \right) \]

- get the original equation back

\[ \csc x = \frac{1}{2}x + 1 \]
\[ \sin x = \frac{1}{2} \]
\[ x = \sin^{-1} \left( \frac{1}{2} \right) \]
\[ x = \frac{1}{2} \]
\[ x = \frac{1}{2} \]
\[ x = \frac{1}{2} \]
\[ x = \frac{1}{2} + 1 \]

Example 5(iv): Use the iterative formula

\[ x_{n+1} = \sin^{-1} \left( \frac{2}{x_n + 2} \right) \]

with initial value \( x_0 = 0.75 \) to determine this root correct to 2 d.p. Give the result of each iteration to 4 d.p.

- use the iteration formula given

At \( x_0 = 0.75 \),
\[ x_1 = 0.8143 \]
\[ x_2 = 0.7864 \]
\[ x_3 = 0.8290 \]
\[ x_4 = 0.7591 \]
\[ x_5 = 0.7867 \]
\[ x_6 = 0.8143 \]
\[ x_7 = 0.7864 \]
\[ x_8 = 0.8290 \]
\[ x_9 = 0.7591 \]
Repetition
Final Ans: \( x = 0.80 \) (2 d.p.)